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W-LIKE MAPS WITH VARIOUS INSTABILITIES OF ACIM'S

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ABSTRACT. This paper generalizes the results of [13] and then provides an interesting example. We construct a family of W-like maps $\{W_a\}$ with a turning fixed point having slope s_1 on one side and $-s_2$ on the other. Each W_a has an absolutely continuous invariant measure μ_a. Depending on whether $\frac{1}{s_1}+\frac{1}{s_2}$ is larger, equal or smaller than 1, we show that the limit of μ_a is a singular measure, a combination of singular and absolutely continuous measure or an absolutely continuous measure, respectively. It is known that the invariant density of a single piecewise expanding map has a positive lower bound on its support. In Section 4 we give an example showing that in general, for a family of piecewise expanding maps with slopes larger than 2 in modulus and converging to a piecewise expanding map, their invariant densities do not necessarily have a positive lower bound on the support.

1. INTRODUCTION

In practice, due to external noise, or roundoff errors in computation, there is a natural interest in the stability of properties of chaotic dynamical systems under small perturbations. If we consider a family of piecewise expanding maps $\tau_a : I \to I$, $a > 0$ with absolutely continuous invariant measures (acim's) μ_a, converging to a piecewise expanding map τ_0 with acim μ_0, then under general assumptions μ_a's converge to μ_0. One such assumption is that $\inf |\tau_a'| > 2$ for all $a > 0$ (see [1], [6], [7] or [10]). This is useful in the study of the metastable systems [15], or to approximate the invariant densities [8].

Keller [9] introduced the family of $\{W_a\}$ maps that are piecewise expanding, ergodic transformations with a "stochastic singularity", i.e., μ_a's converge to a singular measure. This occurs because of the existence of diminishing invariant neighborhoods of the turning fixed point. The slopes of the Keller's W_a maps converge to 2 and -2 on the left and right hand sides of the turning fixed point, respectively.

Given two numbers, s_1 and s_2, greater than 1, we consider a W-like map with one turning fixed point having slope s_1 on one side and $-s_2$ on the other. In [13], the authors considered the special case where $s_1 = s_2 = 2$. Their perturbed maps W_a are piecewise expanding with slopes strictly greater than 2 in modulus and are exact with their acim's supported on all of [0, 1]. The standard bounded variation method [2] cannot be applied in this setting as the slopes of the maps in that family are not uniformly bounded away from 2. Other methods, for example, those studied in [3], [12] and [14] cannot be applied either. Using the main result of [5], it can

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be shown that the μ_a's converge to $\frac{2}{3}\mu_0 + \frac{1}{3}\delta_{(\frac{1}{2})}$, where $\delta_{(\frac{1}{2})}$ is the Dirac measure at point $1/2$ and μ_0 is the acim of the W_0 map. Thus, the family of measures μ_a approach a combination of an absolutely continuous and a singular measure rather than the acim of the limit map. Similar instability was also shown in [4] for a countable family of transitive Markov maps approaching Keller's W_0 map.

In this paper, we construct a family of maps for which the instability of the acim's has a global character, not a local one. In the more general case considered in this paper, with s_1, s_2 not necessarily equal to 2, we will discuss the limits of the acim's μ_a of the $\{W_a\}$ maps. We have three cases:
(I) If $\frac{1}{s_1} + \frac{1}{s_2} > 1$, then μ_a's converge $*$-weakly to $\delta_{(\frac{1}{2})}$.
(II) If $\frac{1}{s_1} + \frac{1}{s_2} = 1$, then μ_a's converge $*$-weakly to

$$\frac{(qs_1 + ps_2 - p - q)(s_2 + 2)}{(qs_1 + ps_2 - p - q)(s_2 + 2) + 2rs_1 s_2^2}\mu_0 + \frac{2rs_1 s_2^2}{(qs_1 + ps_2 - p - q)(s_2 + 2) + 2rs_1 s_2^2}\delta_{(\frac{1}{2})},$$

where p, q and r are parameters defining our family of maps.
(III) If $\frac{1}{s_1} + \frac{1}{s_2} < 1$, then μ_a's converge to μ_0.
Additionally, in Theorem 2, we prove that in case (III) the densities of the μ_a's are uniformly bounded. The first case of our result contains the example in which Keller [9] obtained the "stochastic singularity." In the second case, the limit measure is a combination of an absolutely continuous and a singular measure, and this combination is varying according to p, q and r for fixed s_1 and s_2. This is a generalization of the result of [13]. In the third case, we have a map with a stable acim.

At the end of the paper, we use our main results to provide an interesting example. Keller [11] and Kowalski [12] proved that for a piecewise expanding map $\tau : I \to I$ with $\frac{1}{|\tau'(x)|}$ being a function of bounded variation, the density of the acim of τ has a uniform positive lower bound on its support. We construct a family of piecewise expanding, piecewise linear maps τ_n such that τ_n are exact on $[0,1]$, τ_n converge to $\tau = W_0$ ($s_1 = s_2 = 2$), $|\tau_n'| > 2$ for all n but the densities of the acims μ_n's do not have a uniform positive lower bound.

In Section 2, we introduce our family of W_a maps and state the main result. In Section 3 we present the proofs. In Section 4, we show the example related to the results of Keller [11] and Kowalski [12].

2. Family of W_a maps and the main result

Let s_1, $s_2 > 1$ and p, q, $r > 0$. We consider the family $\{W_a : 0 \le a\}$ of maps of $[0,1]$ onto itself defined by

$$(1) \quad W_a(x) = \begin{cases} 1 - \frac{2(s_1+pa)}{s_1-1+pa-2ra}x \ , & \text{for } \ 0 \le x < \frac{1}{2} - \frac{\frac{1}{2}+ra}{s_1+pa} \ ; \\ (s_1+pa)(x-1/2) + 1/2 + ra \ , & \text{for } \ \frac{1}{2} - \frac{\frac{1}{2}+ra}{s_1+pa} \le x < 1/2 \ ; \\ -(s_2+qa)(x-1/2) + 1/2 + ra \ , & \text{for } \ 1/2 \le x < \frac{1}{2} + \frac{\frac{1}{2}+ra}{s_2+qa} \ ; \\ 1 + \frac{2(s_2+qa)}{s_2-1+qa-2ra}(x-1) \ , & \text{for } \ \frac{1}{2} + \frac{\frac{1}{2}+ra}{s_2+qa} \le x \le 1 \ . \end{cases}$$

For each choice of s_1, $s_2 > 1$, p, q, $r > 0$, we consider only $a > 0$ such that $0 \le W_a(x) \le 1$ for $x \in [0,1]$.

An example of a W_a map is shown in Fig.1. Fig.1(a) is the unperturbed W_0 map with turning fixed point at $1/2$ and $s_1 = 3/2$, $s_2 = 3$. Fig.1(b) is the perturbed map W_a, with $a = 0.05$, $r = 2$, $p = 3$, $q = 2$. The slope of the second branch is

$s_1 + pa = 1.65$, the slope of the third branch is $s_2 + qa = 3.1$, and $W_{0.05}(1/2) = 1/2 + ra = 0.6$.

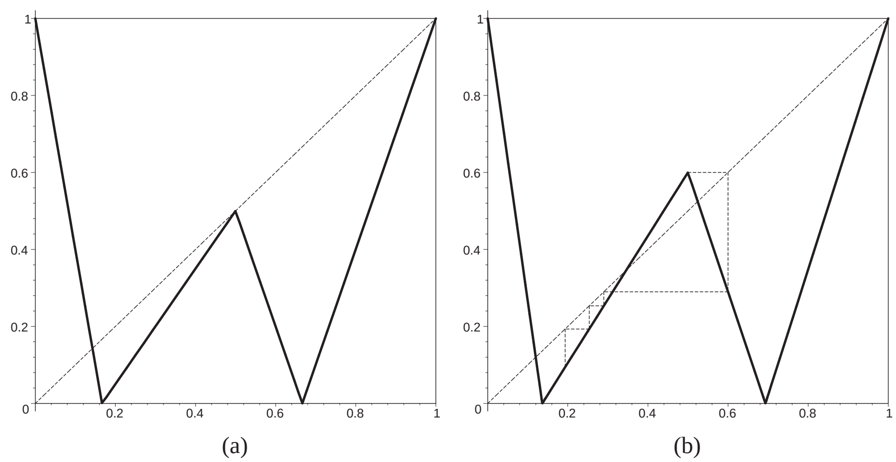

FIGURE 1. The W-like maps with $\frac{1}{s_1} + \frac{1}{s_2} = 1$: (a) W_0 with $s_1 = 3/2$ and $s_2 = 3$, (b) W_a with $s_1 = 3/2$, $s_2 = 3$; $a = 0.05$; $r = 2$, $p = 3$, $q = 2$; also several initial points of the trajectory of $1/2$.

Every W_a has a unique absolutely continuous invariant measure μ_a since all the slopes are greater than 1 in modulus. We will show later that, for $\frac{1}{s_1} + \frac{1}{s_2} \le 1$, μ_a is supported on $[0, 1]$ and for $\frac{1}{s_1} + \frac{1}{s_2} > 1$ it is supported on a subinterval around $1/2$. W_a is an exact map with the measure μ_a. Let h_a denote the normalized density of μ_a, $a \ge 0$. Since the W_0 map is a Markov one, it is easy to check that

$$h_0 = \begin{cases} \frac{2s_1(s_2+1)}{2s_1s_2+s_1-s_2}\ , & \text{for } \ 0 \le x < 1/2\ ; \\ \frac{2s_2(s_1-1)}{2s_1s_2+s_1-s_2}\ , & \text{for } \ 1/2 \le x \le 1\ . \end{cases} \tag{2}$$

Our main result is the following theorem

Theorem 1. *As $a \to 0$ the measures μ_a converge $*$-weakly to the measure*
(I) $\delta_{(\frac{1}{2})}$, if $\frac{1}{s_1} + \frac{1}{s_2} > 1$;
(II) $\frac{(qs_1+ps_2-p-q)(s_2+2)}{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}\mu_0 + \frac{2rs_1s_2^2}{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}\delta_{(\frac{1}{2})}$, if $\frac{1}{s_1} + \frac{1}{s_2} = 1$;
(III) μ_0, if $\frac{1}{s_1} + \frac{1}{s_2} < 1$,
where $\delta_{(\frac{1}{2})}$ is the Dirac measure at point $1/2$.

The proof relies on the general formula for invariant densities of piecewise linear maps [5] and direct calculations. Most objects and quantities we use depend on the parameter a. We suppress a from the notation to make it simpler.

In case (III), we actually prove a little more:

Theorem 2. *If $\frac{1}{s_1} + \frac{1}{s_2} < 1$, then the normalized invariant densities $\{h_a\}$ are uniformly bounded for given p, q and r. Consequently, we obtain Theorem 1(III).*

3. PROOFS

This section contains the proofs of Theorems 1 and 2, divided into a number of steps.

3.1. **Assume $\frac{1}{s_1}+\frac{1}{s_2}>1$.** Let

$$x_l^*=\frac{s_1-1+pa-2ra}{2(s_1-1+pa)}$$

and

$$x_r^*=\frac{s_2s_1-s_2+(2rs_1-q+ps_2+qs_1)a+(2rp+pq)a^2}{2(s_1-1+pa)(s_2+qa)}.$$

x_l^* is the fixed point on the second branch of W_a, and x_r^* is the preimage of x_l^* under the third branch of W_a. Both x_r^* and x_l^* converge to $\frac{1}{2}$ as a approaches 0. For small a, we have

$$W_a(1/2)-x_r^*=\frac{ra\left[s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa)\right]}{(s_1-1+pa)(s_2+qa)}<0.$$

In this case, we have $W_a([x_l^*,x_r^*])\subseteq[x_l^*,x_r^*]$. $W_a|_{[x_l^*,x_r^*]}$ is a skewed tent map with $W_a(1/2)>1/2$; it is known that with acim μ_a, it is exact on $[x_l^*,W_a(1/2)]$. Since μ_a is concentrated on $[x_l^*,x_r^*]$, we conclude that μ_a converge $*$-weakly to $\delta_{(\frac{1}{2})}$. This proves Theorem 1(I).

Fig.2 shows an example with $a=0.05, r=2, p=3, q=2$; $s_1=4/3, s_2=5/2$.

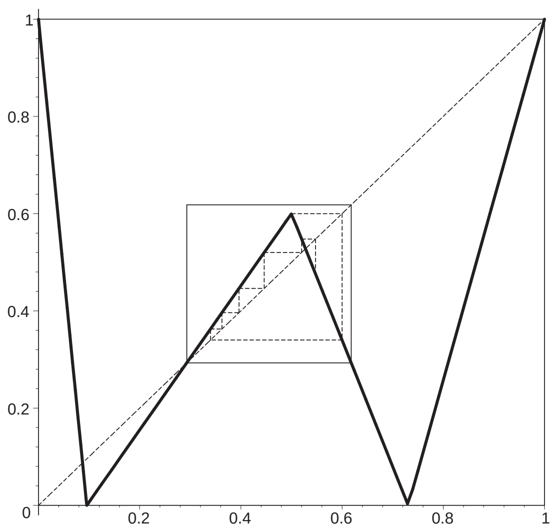

FIGURE 2. The W_a map with $\frac{1}{s_1}+\frac{1}{s_2}>1$

3.2. **Formula for the non-normalized invariant density of W_a if $\frac{1}{s_1}+\frac{1}{s_2}\le 1$.** An example of a map W_a is shown in Fig.1. We have the following proposition.

Proposition 1. *For $\frac{1}{s_1}+\frac{1}{s_2}\le 1$, the map W_a has an absolutely continuous invariant measure μ_a supported on $[0,1]$ and the map W_a with respect to μ_a is exact.*

Proof. W_a is a piecewise expanding transformation. From the general theory (see for example [2]), it follows that it is enough to show that the images $W_a^n(J)$ grow to cover all $[0,1]$ as $n\to\infty$, for any interval $J\subset[0,1]$. Since W_a is expanding, $W_a^n(J)$ grow until some image $W_a^{n_0}(J)$ contains an internal partition point. If this point is not $1/2$, then $W_a^{n_0+2}(J)$ contains the repelling fixed point 1. Then its images grow

to cover all of $[0,1]$. If this point is $1/2$, we proceed as follows. First, assume that $\frac{1}{s_1}+\frac{1}{s_2}<1$. Consider a small neighborhood $J=(z_1,z_2)$ around $1/2$ with length ℓ, then

$$\min_{z_2-z_1=\ell}\max\left\{(\frac{1}{2}-z_1)(s_1+pa),(z_2-\frac{1}{2})(s_2+qa)\right\}=\frac{1}{\frac{1}{s_1+pa}+\frac{1}{s_2+qa}}\ell>\ell.$$

Thus, the interval J will grow until its image covers two partition points of W_a. Then the second iteration afterward will cover $[0,1]$. Therefore, W_a is exact with respect ot μ_a.

Assume $\frac{1}{s_1}+\frac{1}{s_2}=1$. If $a\neq 0$, then $\frac{1}{\frac{1}{s_1+pa}+\frac{1}{s_2+qa}}>1$, which implies W_a is exact with respect to μ_a. In the case $a=0$, we first note that $1/2$ is a turning fixed point. Take again a small interval $J=(z_1,z_2)\ni 1/2$. Its image is an interval $(z,1/2)$. It will grow under iteration and its iterations still contain $1/2$. It will grow until its image covers another partition point of W_a. Then, the second iteration afterward will covers all of $[0,1]$. Thus, W_a is again exact with respect to μ_a. □

We adapt the general formulas of [5] to our case and obtain the following lemma:

Lemma 1. *(I) N=4, K=2, L=0 ;*
(II) $\alpha=(1,1/2+ra,1/2+ra,1)$, $\beta=(\beta_1,\beta_2,\beta_3,\beta_4)$, *where* $\beta_1=-\frac{2(s_1+pa)}{s_1-1+pa-2ra}$, $\beta_2=s_1+pa$, $\beta_3=-(s_2+qa)$ *and* $\beta_4=\frac{2(s_2+qa)}{s_2-1+qa-2ra}$, $\gamma=(0,0,0,0)$ *;*
(III) The digits $A=(a_1,a_2,a_3,a_4)$, *where* $a_1=-1, a_2=\frac{s_1-1+pa-2ra}{2}$, $a_3=-\frac{s_2+1+qa+2ra}{2}$, $a_4=\frac{s_2+1+qa+2ra}{s_1-1+pa-2ra}$ *;*
(IV) There are two c_i*'s, which are* $c_1=(1/2,2)$ *and* $c_2=(1/2,3)$, *and* $j(c_1)=2$, $j(c_2)=3$. *Then,* $W_u=\{c_1,c_2\}$,$W_l=\emptyset$, $U_l=\{c_2\}$,$U_r=\{c_1\}$ *;*
(V) $\beta(c_1,1)=s_1+pa$ *since* $j(c_1)=2$, *then* $\beta(c_1,2)=-(s_1+pa)(s_2+qa)$ *and* $\beta(c_1,k)=-(s_2+qa)(s_1+pa)^{k-1}$ *up to some* k *which is the first moment* j *when the* $W_a^j(1/2)$ *is less than* $\frac{1}{2}-\frac{1/2+ra}{s_1+pa}$, *and is the same one defined in Lemma 4 ;*
(VI) $\beta(c_2,1)=-(s_2+qa)$ *since* $j(c_2)=3$, *then* $\beta(c_2,2)=(s_2+qa)^2$ *and* $\beta(c_2,k)=(s_2+qa)^2(s_1+pa)^{k-2}$ *up to the same* k *in part (e),* $W_a^n(c_1)=W_a^n(c_2)$ *for all* n *;*
(VII) Based on (VI), we have the following for the matrix $S=(S_{i,j})_{1\le i,j\le 2}$ *:*
For $c_1\in U_r$

$$S_{1,1}=\sum_{n=1}^{\infty}\frac{\delta(\beta((c_1,n)>0))\delta(W_a^n(c_1)>1/2)+\delta(\beta((c_1,n)<0))\delta(W_a^n(c_1)<1/2)}{|\beta(c_1,n)|},$$

$$S_{1,2}=\sum_{n=1}^{\infty}\frac{\delta(\beta((c_1,n)>0))\delta(W_a^n(c_1)>1/2)+\delta(\beta((c_1,n)<0))\delta(W_a^n(c_1)<1/2)}{|\beta(c_1,n)|}.$$

For $c_2\in U_l$

$$S_{2,1}=\sum_{n=1}^{\infty}\frac{\delta(\beta((c_2,n)<0))\delta(W_a^n(c_2)>1/2)+\delta(\beta((c_2,n)>0))\delta(W_a^n(c_2)<1/2)}{|\beta(c_2,n)|},$$

$$S_{2,2}=\sum_{n=1}^{\infty}\frac{\delta(\beta((c_2,n)<0))\delta(W_a^n(c_2)>1/2)+\delta(\beta((c_2,n)>0))\delta(W_a^n(c_2)<1/2)}{|\beta(c_2,n)|}.$$

Remark 1. *It follows from (V, VI) of Lemma 1 that*

$$S_{1,1}=S_{1,2}\ ,\ S_{2,1}=S_{2,2}\ \text{and}\ S_{1,1}=\frac{s_2+qa}{s_1+pa}S_{2,2}\ .$$

Let Id be the 2×2 identity matrix and let $V=[1,1]$. Then, for the solution, $D=[D_1,D_2]$, of the system :

$$\left(-S^T+Id\right)D^T=V^T, \tag{1}$$

we have $D_1=D_2$. Let us denote them by Λ.

Let I_1,I_2,I_3,I_4 be the partition of $I=[0,1]$ into maximal intervals of monotonicity of W_a: $I_1=[0,\frac{s_1-1+pa-2ra}{2(s_1+pa)})$, $I_2=(\frac{s_1-1+pa-2ra}{2(s_1+pa)},1/2)$, $I_3=(1/2,\frac{s_2+1+qa+2ra}{2(s_2+qa)})$ and $I_4=(\frac{s_2+1+qa+2ra}{2(s_2+qa)},1]$. We define the following index function:

$$j(x)=j \text{ for } x\in I_j, j=1,2,3,4,$$

and

$$j(c_1)=2, j(c_2)=3.$$

We define the cumulative slopes for iterates of points as follows:

$$\beta(x,1)=\beta_{j(x)},\quad \text{and } \beta(x,n)=\beta(x,n-1)\cdot\beta_{j(W_a^{n-1}(x))},\qquad n\geq 2.$$

In particular, we have

$$\beta(1/2,n)=(s_1+pa)\cdot W_a'(W_a(1/2))\cdot W_a'(W_a^2(1/2))\cdots W_a'(W_a^{n-1}(1/2))\ ,$$

which is the cumulative slope along the n steps of trajectory of $1/2$. Recall that k is the first moment j when the $W_a^j(1/2)$ is less than $\frac{1}{2}-\frac{1/2+ra}{s_1+pa}$. Let $k_1=[\frac{2}{3}k]$ (the integer part of $2k/3$). Note that $k_1\to\infty$ as $a\to 0$. Let

$$\chi^s(t,x)=\begin{cases}\chi_{[0,x]} & \text{for } \ t>0\ ;\\ \chi_{[x,1]} & \text{for } \ t<0\ .\end{cases}$$

Now, we can obtain the following formula for f_a:

Lemma 2. *Let*

$$f_a=1+(1+\frac{s_1+pa}{s_2+qa})\Lambda\left(\sum_{n=1}^{\infty}\frac{\chi^s(\beta(1/2,n),W_a^n(1/2))}{|\beta(1/2,n)|}\right).$$

Then f_a is W_a invariant non-normalized density. Furthermore, for small $a>0$, we have:
(I) If $\frac{1}{s_1}+\frac{1}{s_2}=1$, then $\Lambda<-1$;
(II) If $\frac{1}{s_1}+\frac{1}{s_2}<1$, the sign of Λ depends on s_1 and s_2, can be either positive or negative depending on the sign of $\vartheta=1-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)=1-\frac{s_1+s_2}{s_1s_2}\left(1+\frac{s_1}{s_2(s_1-1)}\right)$. The case when $\vartheta=0$ is discussed at the end of Section 3.

Proof. By the Theorem 2 in [5], it follows from (IV,V,VI) of Lemma 1 that:

$$\begin{aligned} f_a &= 1+D_1\sum_{n=1}^{\infty}\frac{\chi^s(\beta(c_1,n),W_a^n(c_1))}{|\beta(c_1,n)|}+D_2\sum_{n=1}^{\infty}\frac{\chi^s(-\beta(c_2,n),W_a^n(c_2))}{|\beta(c_2,n)|}\\ &= 1+\Lambda\sum_{n=1}^{\infty}\frac{\chi^s(\beta(c_1,n),W_a^n(1/2))}{|\beta(c_1,n)|}+\Lambda\sum_{n=1}^{\infty}\frac{\chi^s(-\beta(c_2,n),W_a^n(1/2))}{|\beta(c_2,n)|}\\ &= 1+(1+\frac{s_1+pa}{s_2+qa})\Lambda\left(\sum_{n=1}^{\infty}\frac{\chi^s(\beta(1/2,n),W_a^n(1/2))}{|\beta(1/2,n)|}\right).\end{aligned}$$

Since

$$S_{1,1} \geq \frac{1}{s_1+pa} + \frac{1}{s_2+qa}\sum_{n=1}^{k_1-1}\frac{1}{(s_1+pa)^n} = \frac{1}{s_1+pa} + \frac{1}{s_2+qa}\frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1} ,$$

$$S_{1,1} \leq \frac{1}{s_1+pa} + \frac{1}{s_2+qa}\sum_{n=1}^{\infty}\frac{1}{(s_1+pa)^n} = \frac{1}{s_1+pa} + \frac{1}{s_2+qa}\frac{1}{s_1+pa-1} ,$$

and $\Lambda = \frac{1}{1-\frac{s_1+s_2+pa+qa}{s_2+qa}S_{1,1}}$, we have

$$\text{(3)} \qquad \Lambda_l = \frac{1}{1-(\kappa+\eta(1-\frac{1}{(s_1+pa)^{k_1-1}}))} \leq \Lambda \leq \frac{1}{1-(\kappa+\eta)} = \Lambda_h ,$$

where $\kappa = \frac{s_1+s_2+pa+qa}{(s_1+pa)(s_2+qa)}$, $\eta = \frac{s_1+s_2+pa+qa}{(s_2+qa)^2(s_1+pa-1)}$.

To obtain the upper bound of $S_{1,1}$, we assume $s_1 < s_2$. For $s_1 > s_2$ the calculations differ slightly.

(I) Note that for small a both estimates Λ_l and Λ_h are smaller than -1 since both κ and η are smaller than 1 and close to 1. Furthermore, as a approaches 0, both κ and η approach 1.

(II) As a approaches 0, κ and η approach $\frac{s_1+s_2}{s_1 s_2}$ and $\frac{s_1+s_2}{s_2^2(s_1-1)}$, respectively. Again, note that for small a, estimates Λ_l and Λ_h can be either positive or negative, and they have the same sign. □

For small positive a, the first image of $1/2$ is $W_a(1/2) = 1/2 + ra$ and the next one falls just below the fixed point x_l^* slightly less than $1/2$. The following images form a decreasing sequence until they go below $\frac{1}{2} - \frac{1/2+ra}{s_1+pa}$. Since k is the first iteration j when the $W_a^j(1/2)$ is less than $\frac{1}{2} - \frac{1/2+ra}{s_1+pa}$, the consecutive cumulative slopes of $1/2$ are

$$(s_1+pa), -(s_1+pa)(s_2+qa), -(s_1+pa)^2(s_2+qa), \ldots, -(s_1+pa)^{k-1}(s_2+qa) ,$$

and

$$\text{(4)} \quad f_a = 1 + (1+\frac{s_1+pa}{s_2+qa})\Lambda\left(\frac{\chi_{[0,W_a(1/2)]}}{(s_1+pa)} + \sum_{j=2}^{k}\frac{\chi_{[W_a^j(1/2),1]}}{(s_1+pa)^{j-1}(s_2+qa)} + \ldots\right) .$$

3.3. **Estimates, normalizations and integrals on f_a for $\frac{1}{s_1}+\frac{1}{s_2} \leq 1$.** Remembering that $k = \min\{j \geq 1 : W_a^j(1/2) \leq \frac{1}{2} - \frac{1/2+ra}{s_1+pa}\}$ and $k_1 = [\frac{2}{3}k]$ (the integer part of $2k/3$), we will give the estimates on f_a.

Let us define

$$g_l = \frac{\chi_{[0,W_a(1/2)]}}{s_1+pa} + \frac{1}{s_2+qa}\sum_{j=2}^{k_1}\frac{\chi_{[W_a^j(1/2),1]}}{(s_1+pa)^{j-1}} ,$$

and

$$g_h = g_l + \frac{1}{s_2+qa}\sum_{j=0}^{\infty}\frac{1}{(s_1+pa)^{j+k_1}} = g_l + \frac{1}{(s_2+qa)(s_1+pa-1)(s_1+pa)^{k_1-1}} .$$

Also, let $\chi_1 = \chi_{[0,1/2+ra]}$, $\chi_j = \chi_{[W_a^j(1/2),1/2+ra]}, j = 2,3,\ldots,k_1$, $\chi_c = \chi_{(1/2+ra,1]}$.

3.3.1. *Estimates on f_a if $\frac{1}{s_1}+\frac{1}{s_2}=1$.* We have the following lemma:

Lemma 3. *For the family of W_a maps, if $\frac{1}{s_1}+\frac{1}{s_2}=1$, we have*
(I) $W_a(1/2)=1/2+ra$, $W_a^2(1/2)=-ra(s_2+qa)+1/2+ra$, and for $3\le m\le k$, we have $W_a^m(1/2)=-a^2(s_1+pa)^{m-2}\frac{r(qs_1+ps_2-p-q)+rpqa}{s_1+pa-1}+\frac{s_1-1+pa-2ra}{2(s_1+pa-1)}$;
(II) $\lim\limits_{a\to 0} ak=0$;
(III) $\lim\limits_{a\to 0}\frac{1}{a(s_1+pa)^k}=0$;
(IV) $\lim\limits_{a\to 0}\frac{1}{a(s_1+pa)^{k_1}}=0$;
(V) $\lim\limits_{a\to 0} a^2(s_1+pa)^{k_1}=0$;
(VI) $\lim\limits_{a\to 0} W_a^{k_1}(\frac{1}{2})=\frac{1}{2}$.

Proof. Suppose (I) is true. Let us first prove that (II) and (III) are true.

By the definition of k, we have:

(5)
$$0\le -a^2(s_1+pa)^{k-2}\frac{r(qs_1+ps_2-p-q)+rpqa}{s_1+pa-1}+\frac{s_1-1+pa-2ra}{2(s_1+pa-1)}\le\frac{1}{2}-\frac{1/2+ra}{s_1+pa}.$$

The first inequality of (5) implies that $(s_1+pa)^{k-2}\le\frac{s_1-1+pa-2ra}{2a^2(r(qs_1+ps_2-p-q)+rpqa)}$, thus

$$ak\le a\frac{\ln(s_1-1+pa-2ra)-\ln 2-2\ln a-\ln(r(qs_1+ps_2-p-q)+rpqa)}{\ln(s_1+pa)}+2a,$$

$$a\le\frac{\sqrt{s_1-1+pa-2ra}(s_1+pa)}{\sqrt{2(r(qs_1+ps_2-p-q)+rpqa)}(s_1+pa)^{k/2}},$$

$$a^2(s_1+pa)^{k_1}\le\frac{(s_1-1+pa-2ra)(s_1+pa)^2}{2(r(qs_1+ps_2-p-q)+rpqa)(s_1+pa)^{k-k_1}},$$

so we obtain (V), and since $\lim\limits_{a\to 0} a\ln a=0$, we obtain (II).

The second inequality of (5) implies

$$\frac{1}{a(s_1+pa)^{k-2}}\le\frac{2a(r(qs_1+ps_2-p-q)+rpqa)(s_1+pa)}{s_1-1+pa-2ra}.$$

Therefore,

$$\frac{1}{a(s_1+pa)^k}\le\frac{2a(r(qs_1+ps_2-p-q)+rpqa)}{(s_1-1+pa-2ra)(s_1+pa)},\tag{6}$$

and as $a\to 0$, we obtain (III).

On the other hand, (6) implies

$$\begin{aligned}\frac{1}{a(s_1+pa)^{k_1}} &\le \frac{2a(r(qs_1+ps_2-p-q)+rpqa)(s_1+pa)^{k-k_1}}{(s_1+pa-2ra-1)(s_1+pa)}\\ &\le \frac{\sqrt{2(r(qs_1+ps_2-p-q)+rpqa)}(s_1+pa)^{k-k_1}}{\sqrt{s_1+pa-2ra-1}(s_1+pa)^{k/2}}\\ &= \frac{\sqrt{2(r(qs_1+ps_2-p-q)+rpqa)}}{\sqrt{s_1+pa-2ra-1}(s_1+pa)^{k_1-k/2}}.\end{aligned}$$

By the definition of k_1, we obtain (IV). (VI) follows from (V).

Now, let us prove (I).

The fixed point slightly less than 1/2 is $x_l^* = \frac{s_1-1+pa-2ra}{2(s_1-1+pa)}$, and

$$x_l^* - W_a^2(1/2) = \frac{ra^2(q(s_1-1)+p(s_2-1)+apq)}{s_1-1+pa} > 0,$$

which implies that $W_a^m(1/2)$ are all in the domain of the second branch of W_a for $3 \le m \le k$. For a linear map $T(x) = m_0 x + b_0$, we have $T^n(x) = m_0^n x + \frac{m_0^n-1}{m_0-1} b_0$. This proves (I). □

Using (4) and (3) we see that for the functions $f_l = 1 + (1 + \frac{s_1+pa}{s_2+qa})\Lambda_l g_h$ and $f_h = 1 + (1 + \frac{s_1+pa}{s_2+qa})\Lambda_h g_l$, we have

$$f_l \le f_a \le f_h \ . \tag{7}$$

Now, we will represent functions f_l and f_c as combinations of functions χ_j, $j = 1, \ldots, k_1$ and χ_c. After some calculations, we obtain

$$\begin{aligned}
f_l \quad &= \quad 1 + (1 + \frac{s_1+pa}{s_2+qa})\Lambda_l \left(\frac{\chi_{[0,W_a(1/2)]}}{s_1+pa} + \frac{1}{s_2+qa} \sum_{j=2}^{k_1} \frac{\chi_{[W_a^j(1/2),1]}}{(s_1+pa)^{j-1}} \right. \\
&\quad \left. + \frac{1}{(s_2+qa)(s_1+pa-1)(s_1+pa)^{k_1-1}} \right) \\
&= \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)} \Lambda_l + 1 \right) \chi_1 + \frac{s_1+s_2+pa+qa}{(s_2+qa)^2} \Lambda_l \sum_{j=2}^{k_1} \frac{\chi_j}{(s_1+pa)^{j-1}} \\
&\quad + \left(\frac{s_1+s_2+pa+qa}{(s_2+pa)^2} \Lambda_l \frac{1 - \frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1} + 1 \right) \chi_c \\
&\quad + \frac{\frac{s_1+s_2+pa+qa}{s_2+qa}\Lambda_l}{(s_2+qa)(s_1+pa-1)(s_1+pa)^{k_1-1}} \ ,
\end{aligned}$$

$$\begin{aligned}
f_h \quad &= \quad 1 + (1 + \frac{s_1+pa}{s_2+qa})\Lambda_h \left(\frac{\chi_{[0,W_a(1/2)]}}{s_1+pa} + \frac{1}{s_2+qa} \sum_{j=2}^{k_1} \frac{\chi_{[W_a^j(1/2),1]}}{(s_1+pa)^{j-1}} \right) \\
&= \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)} \Lambda_h + 1 \right) \chi_1 + \frac{s_1+s_2+pa+qa}{(s_2+qa)^2} \Lambda_h \sum_{j=2}^{k_1} \frac{\chi_j}{(s_1+pa)^{j-1}} \\
&\quad + \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)^2} \Lambda_h \frac{1 - \frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1} + 1 \right) \chi_c \ .
\end{aligned}$$

In the case we are considering, (3) implies that both Λ_l, Λ_h are smaller than -1. Using this, one can show that all the coefficients in the representation of f_l and f_h are negative for sufficiently small a. For example, let us consider the coefficient of χ_1 in f_h:

$$\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)} \Lambda_h + 1 = \frac{\kappa}{1-(\kappa+\eta)} + 1 = \frac{1-\eta}{1-(\kappa+\eta)} < 0 \ .$$

3.3.2. *Normalizations and integrals if* $\frac{1}{s_1}+\frac{1}{s_2}=1$. Let us define $J_1=[0,W_a^{k_1}(1/2)]$, $J_2=(W_a^{k_1}(1/2),1/2+ra]$, $J_3=(1/2+ra,1]$. We will calculate integrals of f_h over each of these intervals J_1, J_2 and J_3, and use them to normalize f_h. We have

$$\begin{aligned}
C_1 &= \int_{J_1} f_h\ d\lambda = \int_{J_1}\left[\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h+1\right]\chi_1\ d\lambda\\
&= \left[\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h+1\right]W_a^{k_1}(\frac{1}{2}) = \left[\frac{\kappa}{1-(\kappa+\eta)}+1\right]W_a^{k_1}(\frac{1}{2})\\
&= \left[\frac{a(2qs_1s_2+ps_2^2-2qs_2-p-q)}{(1-(\kappa+\eta))(s_2+qa)^2(s_1+pa-1)}\right.\\
&\qquad\left.+\frac{a^2(2pqs_2-q^2+q^2s_1)+pq^2a^3}{(1-(\kappa+\eta))(s_2+qa)^2(s_1+pa-1)}\right]W_a^{k_1}(\frac{1}{2})\ .
\end{aligned}$$

Using Lemma 3, we obtain

$$\lim_{a\to 0}\frac{C_1}{a} = -\frac{2qs_1s_2+ps_2^2-2qs_2-p-q}{2s_2^2(s_1-1)} = -\frac{2qs_1+ps_2^2-p-q}{2s_2s_1}.$$

In the same way, we can see that for any $0<\theta<1/2$, we obtain

$$\lim_{a\to 0}\frac{1}{a}\int_0^{\theta} f_h d\lambda = -\frac{2qs_1+ps_2^2-p-q}{s_2s_1}\theta\ .$$

On the interval J_2, the integral of f_h is:

$$\begin{aligned}
C_2=\int_{J_2} f_h\ d\lambda &= \int_{J_2}\left[\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h+1\right]\chi_1\ d\lambda\\
&\quad+\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\sum_{j=2}^{k_1}\int_{J_2}\frac{\chi_j}{(s_1+a)^{j-1}}\ d\lambda\\
&= \frac{1-\eta}{1-(\kappa+\eta)}\left(\frac{1}{2}+ra-W_a^{k_1}(\frac{1}{2})\right)\\
&\quad+\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\left[\frac{ra(s_2+qa)}{s_1+pa}+\frac{ra(1-\frac{1}{(s_1+pa)^{k_1-2}})}{(s_1+pa-1)^2}\right.\\
&\quad\left.+\frac{a^2(k_1-2)}{s_1+pa}\frac{r(qs_1+ps_2-p-q)+rpqa}{s_1+pa-1}\right]\ .
\end{aligned}$$

Using Lemma 3, we obtain

$$\lim_{a\to 0}\frac{C_2}{a} = -\frac{s_1+s_2}{s_2^2}\left[\frac{rs_2}{s_1}+\frac{r}{(s_1-1)^2}\right] = -rs_2.$$

On the interval J_3, the integral of f_h is:

$$\begin{aligned}
C_3=\int_{J_3} f_h\ d\lambda &= \int_{J_3}\left(\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1}+1\right)\chi_c\ d\lambda\\
&= \left[\left(1-\frac{1}{(s_1+pa)^{k_1-1}}\right)\frac{\eta}{1-(\kappa+\eta)}+1\right](\frac{1}{2}-ra)\\
&= \frac{\frac{a(qs_1+ps_2-p-q)+pqa^2}{(s_1+pa)(s_2+qa)}-\frac{\eta}{(s_1+pa)^{k_1-1}}}{1-(\kappa+\eta)}(\frac{1}{2}-ra)\ .
\end{aligned}$$

Using Lemma 3, we obtain

$$\lim_{a\to 0}\frac{C_3}{a}=-\frac{qs_1+ps_2-p-q}{2s_1s_2}\ .$$

In the same way, we can see that for any $0<\theta<1/2$, we obtain

$$\lim_{a\to 0}\frac{1}{a}\int_{1/2+\theta}^{1} f_h d\lambda=-\frac{qs_1+ps_2-p-q}{s_1s_2}\left(\frac{1}{2}-\theta\right)\ .$$

If we define $B=C_1+C_2+C_3$, then $\frac{f_h}{B}$ is a normalized density. We see that

$$\lim_{a\to 0}\frac{B}{a}=-\frac{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}{2s_1s_2}\ .$$

Our calculations show that the normalized measures $\{(f_h/B)\cdot\lambda\}$ converge $*$-weakly to the measure

$$\frac{(qs_1+ps_2-p-q)(s_2+2)}{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}\mu_0+\frac{2rs_1s_2^2}{(qs_1+ps_2-p-q)(s_2+2)+2rs_1s_2^2}\delta_{(\frac{1}{2})}\ .$$

Now, we will show the same holds for the normalized measure defined by f_l. To this end, let us notice that

$$\begin{aligned}
f_h-f_l &= (1+\frac{s_1+pa}{s_2+qa})\Lambda_h g_l-(1+\frac{s_1+pa}{s_2+qa})\Lambda_l g_h \\
&= (1+\frac{s_1+pa}{s_2+qa})(\Lambda_h-\Lambda_l)g_l-\Lambda_l\frac{1+\frac{s_1+pa}{s_2+qa}}{(s_2+qa)(s_1+pa-1)(s_1+pa)^{k_1-1}} \\
&= (1+\frac{s_1+pa}{s_2+qa})\frac{\frac{\eta}{(s_1+pa)^{k_1-1}}}{[1-(\kappa+\eta)][1-\kappa-\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})]}g_l \\
&\quad -\Lambda_l\frac{1+\frac{s_1+pa}{s_2+qa}}{(s_2+qa)(s_1+pa-1)(s_1+pa)^{k_1-1}}\ ,
\end{aligned}$$

where $|g_l|\le\frac{2}{s_1}$ and $\lim_{a\to 0}\Lambda_l=-1$. Using Lemma 3 once again, we can show that for any subinterval $J\subset[0,1]$, we have

$$\lim_{a\to 0}\frac{1}{a}\int_J(f_h-f_l)d\lambda=0\ .$$

For $J=[0,1]$ this means that the normalizations of f_l and f_h are asymptotically the same. With this, the limit for a general J means in particular that the $*$-weak limit of normalized measures defined using f_l is the same as for those defined using f_h. In view of inequality (7), this proves Theorem 1(II).

3.3.3. *Estimates on f_a if $\frac{1}{s_1}+\frac{1}{s_2}<1$.* We have the following lemma:

Lemma 4. *For the family of W_a maps, if $\frac{1}{s_1}+\frac{1}{s_2}<1$, we have*
(I) $W_a(1/2)=1/2+ra$, $W_a^2(1/2)=-ra(s_2+qa)+1/2+ra$, and for $3\le m\le k$, we have $W_a^m(1/2)=-a(s_1+pa)^{m-2}\frac{r[s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa)]}{s_1+pa-1}+\frac{s_1-1+pa-2ra}{2(s_1+pa-1)}$;
(II) $\lim_{a\to 0}ak=0$;
(III) $\lim_{a\to 0}a(s_1+pa)^{k_1}=0$;
(IV) $\lim_{a\to 0}W_a^{k_1}(\frac{1}{2})=\frac{1}{2}$.

Proof. Suppose (I) is true. Let us first prove that (II) and (III) are true.

By the definition of k, we have:

$$
\begin{aligned}
0 \leq &- a(s_1+pa)^{k-2}\frac{r\left[s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa)\right]}{s_1+pa-1} \\
&+ \frac{s_1-1+pa-2ra}{2(s_1+pa-1)}.
\end{aligned} \tag{8}
$$

The inequality (8) implies $a(s_1+pa)^{k-2} \leq \frac{s_1-1+pa-2ra}{2r[s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa)]}$, thus

$$
\begin{aligned}
ak \leq \quad & a\frac{\ln(s_1-1+pa-2ra)-\ln 2+2\ln(s_1+pa)-\ln r-\ln a}{\ln(s_1+pa)} \\
& -a\frac{\ln(2r\left[s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa)\right])}{\ln(s_1+pa)}, \\
a(s_1+pa)^{k_1} \leq \quad & \frac{(s_1-1+pa-2ra)(s_1+pa)^2}{2r\left[s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa)\right](s_1+pa)^{k-k_1}},
\end{aligned}
$$

and since $\lim\limits_{a\to 0} a\ln a = 0$, we obtain (II) and (III). (IV) follows from (III).

Now, let us prove (I).

The fixed point slightly less than 1/2 is $x_l^* = \frac{s_1-1+pa-2ra}{2(s_1-1+pa)}$, and

$$
x_l^* - W_a^2(1/2) = \frac{ra\left[s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa)\right]}{s_1-1+pa} > 0,
$$

which implies that $W_a^m(1/2)$ are all in the domain of the second branch of W_a for $3 \leq m \leq k$. Now, (I) follows by the same reasoning as in Lemma 3. □

Lemma 5. *If the normalized densities $\{h_a\}_{a<a_0}$, for some $a_0>0$, are uniformly bounded, then $h_a \to h_0$ in L^1.*

Proof. The uniform boundedness implies $\{h_a\}_{a<a_0}$ is a weakly precompact set in L^1. Thus, any limit of $\{h_a\}_{a<a_0}$ is a invariant density by Proposition 11.3.1 [2]. At the same time, this limit is an L^1 function, thus defines an absolutely continuous invariant measure. Since the map W_0 is exact and has only one acim, we conclude that $h_a \to h_0$ in L^1. □

Now, we will prove Theorem 2:

The main idea of the proof is the following: since non-normalized densities $\{f_a\}$ are uniformly bounded (formulas (9, 10, 11)), it is enough to show that $\{\int_0^1 f_a \, d\lambda\}$ are uniformly separated from zero.

For small a, by Lemma 2, Λ (and then both Λ_l and Λ_h) can be either positive or negative. Thus, we can have the following cases.

Case (i): $\Lambda_l < 0$:

Comparing with (4) and (3), we see that for the functions $\widehat{f}_l = 1+(1+\frac{s_1+pa}{s_2+qa})\Lambda_l g_h$ and $\widehat{f}_h = 1+(1+\frac{s_1+pa}{s_2+qa})\Lambda_h g_l$, we have

$$
\widehat{f}_l \leq f_a \leq \widehat{f}_h \ . \tag{9}
$$

Note that $\widehat{f}_l$ and $\widehat{f}_h$ have the same form as f_l and f_h in Section 3.3.1, so their representations as combinations of functions χ_j, $j=1,\ldots,k_1$ and χ_c are similar to

that of f_l and f_h. At the same time, now we have $\frac{1}{s_1}+\frac{1}{s_2}<1$, so the representation is as follows:

$$\begin{aligned}\widehat{f_l} &= \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_l+1\right)\chi_1+\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_l\sum_{j=2}^{k_1}\frac{\chi_j}{(s_1+pa)^{j-1}}\\ &+\left(\frac{s_1+s_2+pa+qa}{(s_2+pa)^2}\Lambda_l\frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1}+1\right)\chi_c\\ &+\frac{\frac{s_1+s_2+pa+qa}{s_2+qa}\Lambda_l}{(s_2+qa)(s_1+pa-1)(s_1+pa)^{k_1-1}}\ ,\end{aligned}$$

$$\begin{aligned}\widehat{f_h} &= \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h+1\right)\chi_1+\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\sum_{j=2}^{k_1}\frac{\chi_j}{(s_1+pa)^{j-1}}\\ &+\left(\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1}+1\right)\chi_c\ .\end{aligned}$$

(3) implies that all the coefficients in the representation of $\widehat{f_l}$ and $\widehat{f_h}$ are negative for sufficiently small a.

We use the same notations J_1, J_2 and J_3 as in Section 3.3.2. First, we do the calculations assuming that $\vartheta = 1-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)\neq 0$.

We will calculate the integrals of $\widehat{f_h}$ over each of J_1, J_2 and J_3, and use them to normalize $\widehat{f_h}$. We have

$$\begin{aligned}\widehat{C}_1 &= \int_{J_1}\widehat{f_h}\ d\lambda=\int_{J_1}\left[\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h+1\right]\chi_1\ d\lambda\\ &= \left[\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h+1\right]W_a^{k_1}(\frac{1}{2})=\left[\frac{\kappa}{1-(\kappa+\eta)}+1\right]W_a^{k_1}(\frac{1}{2})\\ &= \left[\frac{s_1s_2^2-s_1-s_2-s_2^2}{1-(\kappa+\eta))(s_2+qa)^2(s_1+pa-1)}\right.\\ &\quad+\frac{a(2qs_1s_2+ps_2^2-2qs_2-p-q)}{(1-(\kappa+\eta))(s_2+qa)^2(s_1+pa-1)}\\ &\quad\left.+\frac{a^2(2pqs_2-q^2+q^2s_1)+pq^2a^3}{(1-(\kappa+\eta))(s_2+qa)^2(s_1+pa-1)}\right]W_a^{k_1}(\frac{1}{2})\ .\end{aligned}$$

Using Lemma 4, we have

$$\lim_{a\to 0}\widehat{C}_1=\frac{1}{2}\frac{\frac{s_1s_2^2-s_1-s_2-s_2^2}{s_2^2(s_1-1)}}{1-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)}=\frac{1}{2}\frac{1-\frac{s_1+s_2}{s_2^2(s_1-1)}}{1-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)}.$$

On the interval J_2, the integral of $\widehat{f_h}$ is:

$$\begin{aligned}\widehat{C}_2=\int_{J_2}\widehat{f_h}\ d\lambda &= \int_{J_2}\left[\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)}\Lambda_h+1\right]\chi_1\ d\lambda\\ &+\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\sum_{j=2}^{k_1}\int_{J_2}\frac{\chi_j}{(s_1+pa)^{j-1}}\ d\lambda\end{aligned}$$

$$
\begin{aligned}
&= \frac{1-\eta}{1-(\kappa+\eta)}\left(\frac{1}{2}+ra-W_a^{k_1}(\frac{1}{2})\right) \\
&\quad +\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\Bigg[\frac{ra(s_2+qa)}{s_1+pa}+\frac{ra(1-\frac{1}{(s_1+pa)^{k_1-2}})}{(s_1+pa-1)^2} \\
&\quad +\frac{a(k_1-2)}{s_1+pa}\frac{r(s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q+pqa))}{s_1+pa-1}\Bigg] .
\end{aligned}
$$

Using Lemma 4, we have $\lim_{a\to 0}\widehat{C}_2=0$.

On the interval J_3, the integral of $\widehat{f}_h$ is:

$$
\begin{aligned}
\widehat{C}_3=\int_{J_3}\widehat{f}_h\,d\lambda &= \int_{J_3}\left(\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\Lambda_h\frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1}+1\right)\chi_c\,d\lambda \\
&= \left[\left(1-\frac{1}{(s_1+pa)^{k_1-1}}\right)\frac{\eta}{1-(\kappa+\eta)}+1\right](\frac{1}{2}-ra) \\
&= \frac{\frac{s_1s_2-s_1-s_2+a(qs_1+ps_2-p-q)+pqa^2}{(s_1+pa)(s_2+qa)}-\frac{\eta}{(s_1+pa)^{k_1-1}}}{1-(\kappa+\eta)}(\frac{1}{2}-ra) .
\end{aligned}
$$

Using Lemma 4 once again, we have

$$
\lim_{a\to 0}\widehat{C}_3=\frac{1}{2}\frac{1-\frac{s_1+s_2}{s_1s_2}}{1-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)} .
$$

Note that if we define $\widehat{B}=\widehat{C}_1+\widehat{C}_2+\widehat{C}_3$, then

$$
\lim_{a\to 0}\widehat{B}=\frac{1}{2}\frac{2-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)}{1-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)} ,
$$

which is not 0. Since $\{\widehat{f}_h\}$ are uniformly bounded, we conclude that the normalized $\{\widehat{f}_h\}$ are also uniformly bounded.

Now, we will show that the normalized $\{\widehat{f}_l\}$ are also uniformly bounded. To this end, let us notice that

$$
\begin{aligned}
\widehat{f}_h-\widehat{f}_l &= (1+\frac{s_1+pa}{s_2+qa})\Lambda_h g_l-(1+\frac{s_1+pa}{s_2+qa})\Lambda_l g_h \\
&= (1+\frac{s_1+pa}{s_2+qa})(\Lambda_h-\Lambda_l)g_l-\Lambda_l\frac{1+\frac{s_1+pa}{s_2+qa}}{(s_2+qa)(s_1+pa-1)(s_1+pa)^{k_1-1}} \\
&= (1+\frac{s_1+pa}{s_2+qa})\frac{\frac{\eta}{(s_1+pa)^{k_1-1}}}{[1-(\kappa+\eta)][1-\kappa-\eta(1-\frac{1}{(s_1+pa)^{k_1-1}})]}g_l \\
&\quad -\Lambda_l\frac{1+\frac{s_1+pa}{s_2+qa}}{(s_2+qa)(s_1+pa-1)(s_1+pa)^{k_1-1}} ,
\end{aligned}
$$

where $|g_l|\le\frac{1}{s_1}+\frac{1}{s_2(s_1-1)}$ and $\lim_{a\to 0}\Lambda_l=\frac{1}{1-\left(\frac{s_1+s_2}{s_1s_2}+\frac{s_1+s_2}{s_2^2(s_1-1)}\right)}$. Thus, $\lim_{a\to 0}\widehat{f}_h-\widehat{f}_l=0$.

We conclude that the normalized $\{\widehat{f}_l\}$ are uniformly bounded since the normalized $\{\widehat{f}_h\}$ are uniformly bounded. Thus, after normalization, $\{f_a\}$ are also uniformly bounded.

Case (ii): $\Lambda_l > 0$:

This case implies that f_a given by (4) has the following properties:

$$f_a \geq 1 \ , \tag{10}$$

and all the coefficients of the characteristic functions appearing in (4) are positive. We note that Λ is always positive for small a. Thus,

$$f_a \leq 1 + (1 + \frac{s_1 + pa}{s_2 + qa})\Lambda \sum_{n=1}^{\infty} \frac{1}{|\beta(1/2, n)|} \ , \tag{11}$$

which is finite since our maps $\{W_a\}$ are expanding. In view of (10), we conclude that the normalized $\{f_a\}$ are uniformly bounded.

If $\vartheta = 1 - \left(\frac{s_1+s_2}{s_1 s_2} + \frac{s_1+s_2}{s_2^2(s_1-1)}\right) = 0$, then we have $\lim\limits_{a\to 0} \frac{1}{\Lambda_l} = \lim\limits_{a\to 0} \frac{1}{\Lambda_h} = 0$, Λ_l and Λ_h are still of the same sign. We can renormalize f_a. Let us take the $\widehat{f}_h$ as an example. Multiplying it by $\frac{1}{\Lambda_h}$, we obtain

$$\begin{aligned} \frac{1}{\Lambda_h}\widehat{f}_h \quad &= \quad \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)(s_1+pa)} + \frac{1}{\Lambda_h}\right)\chi_1 + \frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\sum_{j=2}^{k_1}\frac{\chi_j}{(s_1+pa)^{j-1}} \\ &\quad + \left(\frac{s_1+s_2+pa+qa}{(s_2+qa)^2}\frac{1-\frac{1}{(s_1+pa)^{k_1-1}}}{s_1+pa-1} + \frac{1}{\Lambda_h}\right)\chi_c \ . \end{aligned}$$

Note that the coefficients of χ_1 and χ_c converge to $\frac{s_1+s_2}{s_1 s_2}$ and $\frac{s_1+s_2}{s_2^2(s_1-1)}$, respectively. Thus, $\{\int_0^1 \frac{1}{\Lambda_h}\widehat{f}_h \ d\lambda\}$ are separated from 0. This implies $\{\frac{1}{\Lambda_h}\widehat{f}_h\}$ are uniformly bounded. A similar procedure can be applied to $\widehat{f}_l$. We conclude that $\{\frac{1}{\Lambda}f_a\}$ are uniformly bounded.

4. Example

One of the important properties of a piecewise expanding transformation of an interval is that its invariant density is bounded away from 0 on its support. The following result was proved, by Keller [11] and by Kowalski [12].

Theorem 3. *Let a transformation $\tau : I \to I$ be piecewise expanding with $\frac{1}{|\tau'(x)|}$ a function of bounded variation, and let f be a τ-invariant density which can be assumed to be lower semicontinuous. Then there exists a constant $c > 0$ such that $f|_{\text{supp } f} > c$.*

We provide an example showing that this result cannot be generalized to a family of expanding maps, even if they all have this property and converge to a limit map also with this property. Let $d(\cdot,\cdot)$ be the metric on the weak topology of measures.

Example 1. *Let us fix*

$$s_1 = s_2 = 2, \ p = q = 1.$$

For small $a > 0$, let $W_{a,r}$ denote the W_a maps with varying parameter r, and let $\mu_{a,r}$ denote the absolutely continuous invariant measure of $W_{a,r}$. We know that $\mu_{a,r}$ is supported on $[0,1]$ and $W_{a,r}$ with $\mu_{a,r}$ is exact. Using Theorem 1, we know that $\{\mu_{a,r}\}$ converge $*$-weakly to the measure

$$\mu_{0,r} = \frac{1}{1+2r}\mu_0 + \frac{2r}{1+2r}\delta_{\frac{1}{2}}.$$

Let $r_n = n$, $n = 1, 2, 3, \cdots$. Also, let $\{a_n\}_1^\infty$ satisfy $r_n a_n < 1/2$ and be so small that

$$d(\mu_{a_n,r_n}, \mu_{0,r_n}) < \frac{1}{n} \ .$$

Now, for the family of maps $\tau_n = W_{a_n,r_n}$, $n = 1, 2, 3, \cdots$, τ_n converge to W_0 with $|\tau_n'(x)| > 2$, but the invariant densities μ_{a_n,r_n} converge to $\delta_{(\frac{1}{2})}$. This implies that the invariant densities $\{f_{a_n,r_n}\}$ corresponding to $\{\mu_{a_n,r_n}\}$ have no uniform positive lower bound.

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